8.EE

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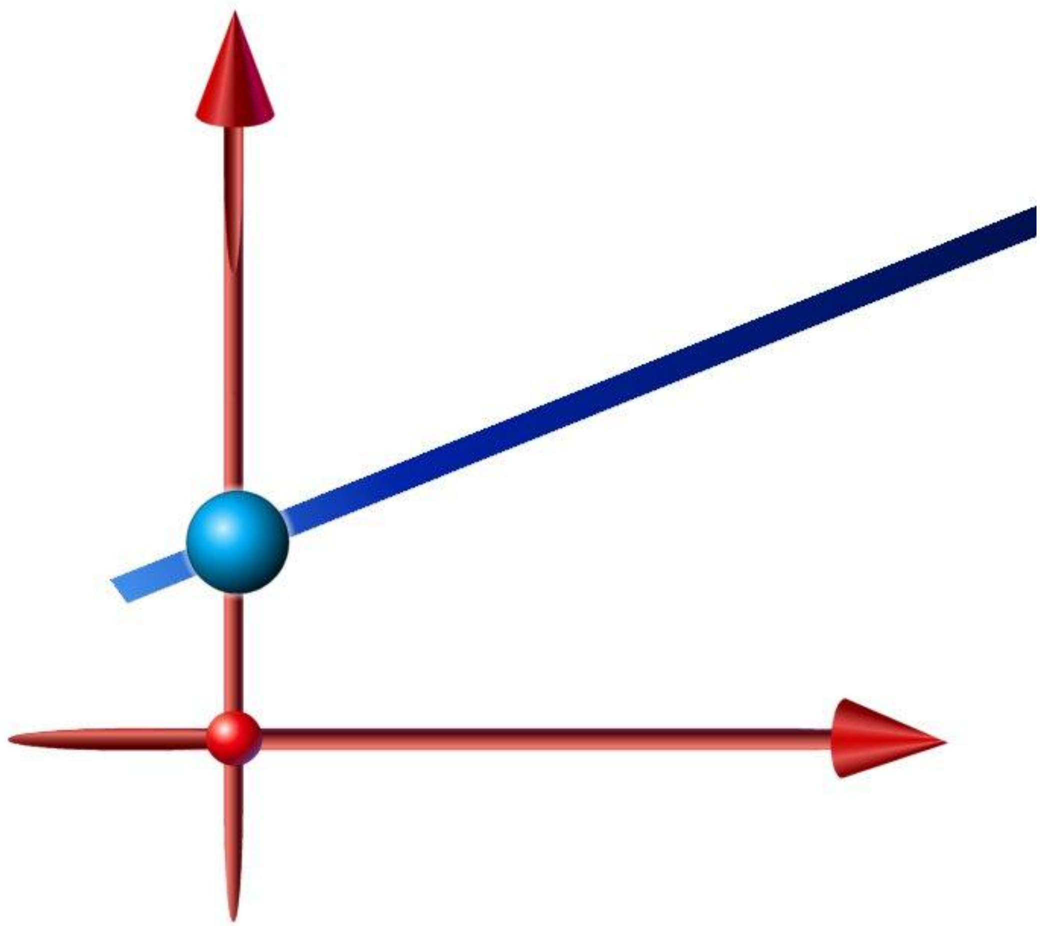

TEACHER'S GUIDE

8.EE.6 DERIVING EQUATIONS FOR LINES WITH NON-ZERO Y-INTERCEPTS

Development from y = mx to y = mx + b

DRAFT 2012.11.29

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Lesson 5: Deriving Equations for Lines with Non-Zero *y*-Intercepts *y* = *mx* + *b*

I. Overview

Essential Questions to be addressed in the lesson

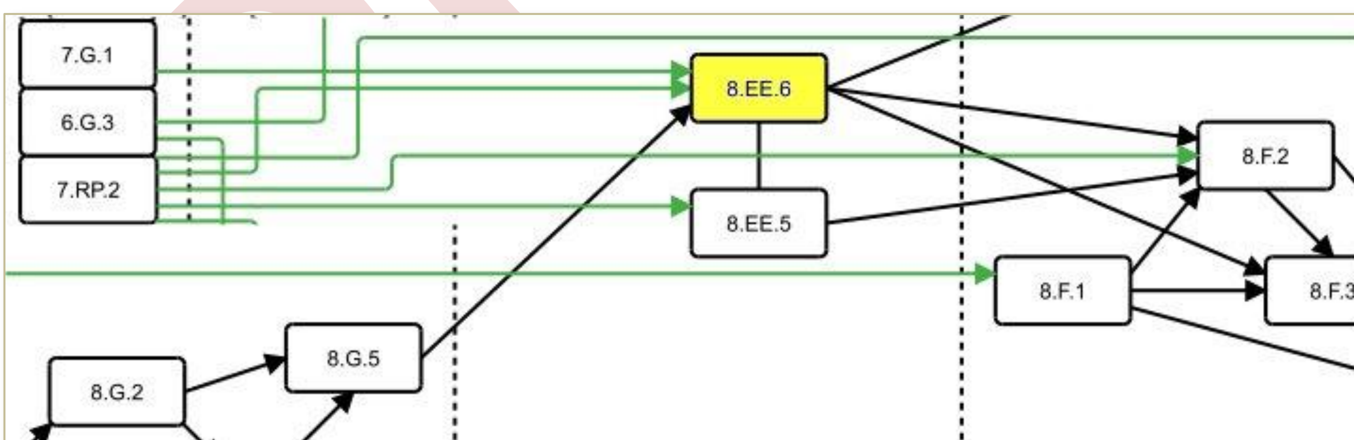
How to develop y = mx + b from y = mx?

Standards to be addressed in this lesson

8.EE.6 Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; **derive the equation y = mx for a line through the origin and the equation y = mx + b for a line intercepting the vertical axis at b.**

According to the Common Core Standards Map, 8.EE.6, 8.EE.5, and 8.F.2 are inherently connected and should be reflected in the teaching sequence.

Figure 1.



Two additional standards strongly related to this standard are:

> **8.EE.5** Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.
>
> This standard will be addressed in *Advanced Content Knowledge* prior to this lesson.

> **8.F.2** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear functions represented by an algebraic expression, determine which function has the greater rate of change.*
>
> This standard will be addressed indirectly in this lesson and directly in a future lesson.

Standards of Mathematical Practice

Although all eight Standards of Mathematical Practice should be instilled in students in these topics, three of them were chosen to be highlighted with these symbols: MPX.

- MP1: **Make sense of problems and persevere in solving them.**
- MP4: **Model with mathematics**.
- MP6: **Attend to precision.**

The three components of *rigor* in the Common Core Standards (Computation Fluency, Conceptual Understanding and Problem Solving) will be denoted by Fluency Concept Application . ***Additional materials should be used to make the Rigor and Mathematical Practice Standards come alive. (For example, see a separate document Challenging Problems and Tasks.)***

II. Advanced Content Knowledge

1. How do I make the connections between proportional relationships, lines and linear equations? Concept

As students in Grade 8 move towards an understanding of the idea of a function, they begin to tie together a number of notions that have been developing over the last few grades:

1. An expression in one variable defines a general calculation in which the variable can represent a range of numbers—an input-output machine with the variable representing the input and the expression calculating the output. For example, $60t$ is the distance traveled in t hours by a car traveling at a constant speed of 60 miles per hour.

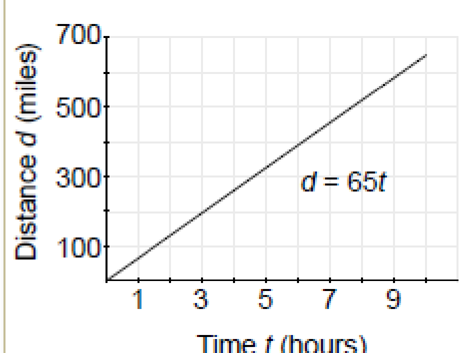
2. Choosing a variable to represent the output leads to an equation in two variables describing the relation between two quantities. For example, choosing d to represent the distance traveled by the car traveling at 65 miles per hour yields the equation $d = 65t$.

3. Tabulating values of the expression is the same as tabulating solution pairs of the corresponding equation. This gives insight into the nature of the relationship; for example, that distance increases by the same amount for the same increase in the time (the ratio between the two being the speed).

4. Plotting points on the coordinate plane, in which each axis is marked with a scale representing one quantity, affords a visual representation of the relationship between two quantities.

Figure 2:

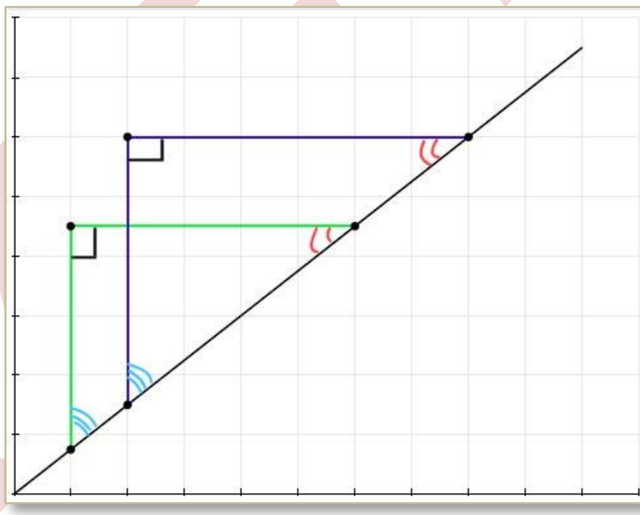
t (hours)	1	2	3	4	5	6
$60t$ (miles)	60	120	180	240	300	360



Proportional relationships provide an opportunity in which these notions can grow together. The constant of proportionality is visible in each; as the multiplicative factor in the expression, as the slope of the line, and as an increment in the table (if the independent variable goes up by 1 unit in each entry.) As students start to build a unified notion of the concept of function they are able to compare proportional relationships presented in different formats. For example, the table shows 300 miles in 5 hours for one vehicle, whereas the graph shows more than 300 miles in the same time for another vehicle. See figure 2. [8.EE.5] In the language of functions, although the distance-time relationship is represented in different ways—one graphically and the other in a table—students are learning to compare the rates and determine that the function represented in the graph has a greater rate of change then the function represented in the table. [8.F.2]

2. Why does a straight line have a constant slope, no matter which two points are chosen to calculate the slope? Concept

The fact that a straight (non-vertical) line has a constant well-defined slope—that the ratio between the vertical change and the horizontal change for *any* two points on the line is always the same—can be explained using similar triangles.[8.EE.6]



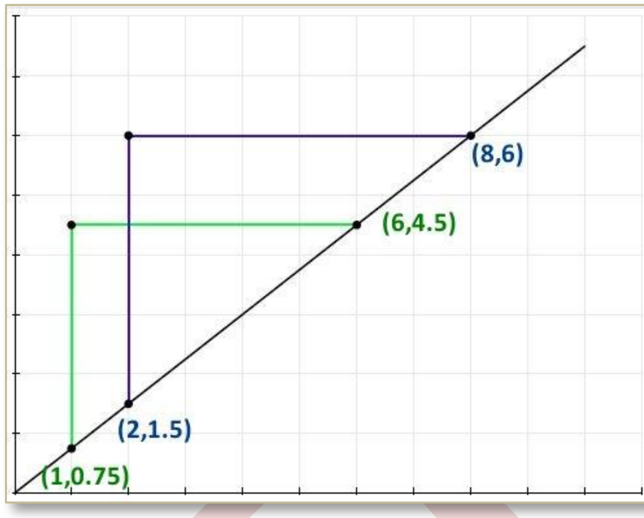
In the graph to the right, two different triangles are chosen to calculate the slope of the line—the blue (bigger one) and the green (smaller one). The horizontal segments of the two triangles are parallel; so are the vertical segments. The line can be viewed as a transversal intersecting these two sets of parallel line segments. The two angles marked with two red ticks are corresponding angles; they are congruent. The two angles marked with three blue ticks are also corresponding angles; they are congruent. Each triangle also has a right angle. Therefore the two triangles are similar (AAA).

In similar triangles, the corresponding sides are proportional. That means the ratio of vertical change and horizontal change for each of the two triangles are equal. Therefore the slope of the line is a constant everywhere, and its value can be calculated by choosing any two different points on the line.

The connection between the unit rate in a proportional relationship and the slope of its graph can be made explicit using real life examples.

The slope of the line can be found by finding the ratio of the vertical change to the horizontal change. Using the two points associated with the green (smaller) triangle,

$$m = \frac{6-1}{4.5-0.75} = \frac{5}{3.75} = \frac{4}{3}$$



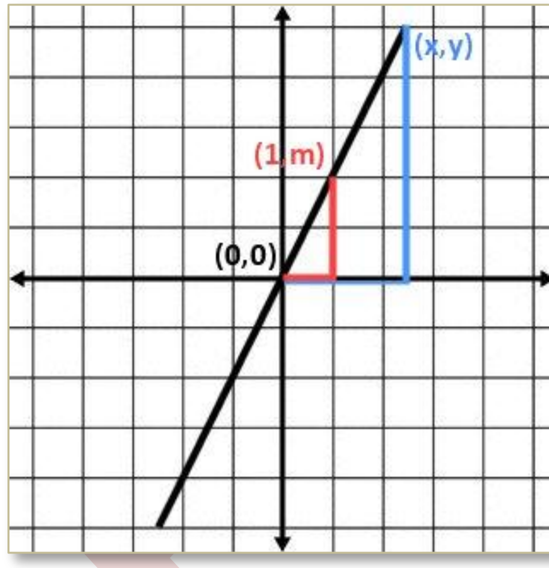
Using the two points associated with the green (larger) triangle

$$m = \frac{8-2}{6-1.5} = \frac{6}{4.5} = \frac{4}{3}$$

The calculated slopes are indeed identical, independent of the points chosen.

3. Where does the generic form of linear equations *y* = *mx* + *b* come from? Can it be derived conceptually? Concept

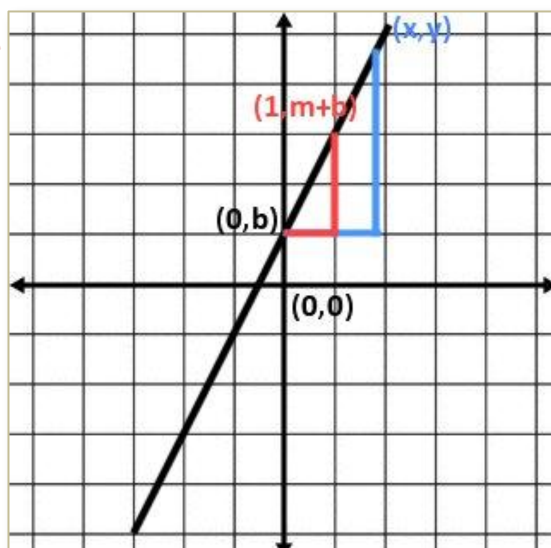
Yes. The very fact that the slope is constant between *any* two points on a line offers a clean way to derive an equation for the line. Here is a demonstration of how to derive linear equations in slope-intercept form.



For a line through the origin, the right triangle whose hypotenuse is the line segment from (0,0) to *any* point (x,y) on the line is similar to the right triangle from (0,0) to the point (1,m) on the line. Using the slope as the vertical change over the horizontal change for the segment (0,0) to (x,y), $\frac{y-0}{x-0}=\frac{y}{x}$. This slope is equal to the slope calculate between the points (0,0) and (1,m), $\frac{m-0}{1-0}=\frac{m}{1}$. Equating these two slopes and solving for y produces the equation of a line that travels through the origin.

$$\frac{y}{x}=\frac{m}{1}\text{, or} \qquad y = mx.$$

The equation for a line not through the origin can be derived in a similar way, starting from the *y*-intercept (0, b) instead of the origin.



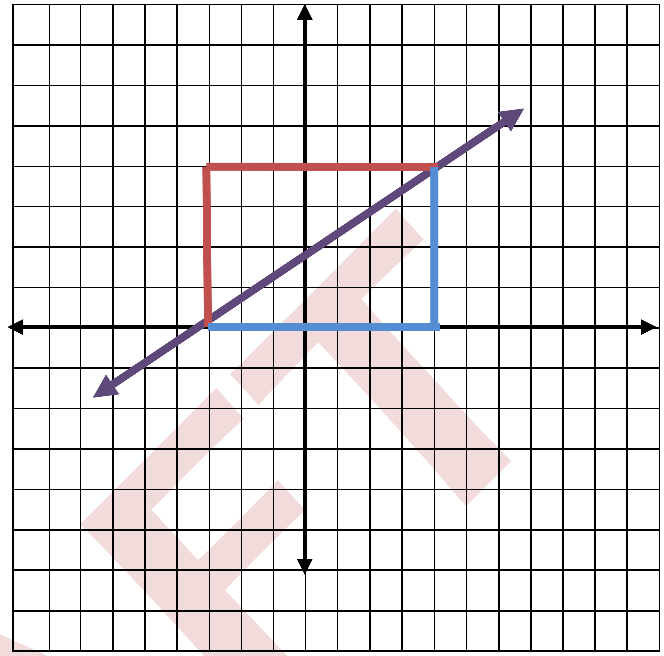
The right triangle whose hypotenuse is the line segment from (0, b) to *any* point (x,y) on the line is similar to the right triangle from (0, b) to the point (1,$m+b$) on the line. Using the slope as the vertical change over the horizontal change for the segment (0, b) to (x,y), $\frac{y-b}{x-0}=\frac{y-b}{x}$. This slope is equal to the slope calculate between the points (0,b) and (1,$m+b$), $\frac{m+b-b}{1-0}=\frac{m}{1}$. Equating these two slopes and solving for y produces the equation of a line that travels through the origin.

$$\frac{y-b}{x}=\frac{m}{1}\text{, or} \quad y-b=mx\text{, or} \qquad y=mx+b.$$

4. When I find my slope graphically, does how I draw in the right triangle impact the slope?

No. It does not matter which of the two triangles one uses.

In the figure to the right, two right triangles are constructed: one above the line (red) and the other below (blue). Does it matter which triangle one uses as the visual guide when finding a slope graphically?



Calculating the slope of the upper (red) triangle, the vertical change is 4 and the horizontal change is 7, which gives the line a slope of $\frac{4}{7}$. Calculating the slope of the lower (blue) triangle, the vertical change is still 4 and the horizontal change is still 7, which gives the same slope of $\frac{4}{7}$.

5. Does which point I assign as (x_1, y_1), and which I assign as (x_2, y_2) affect the slope?

Does the exchange of (x_1, y_1) and (x_2, y_2) affect the calculated value of the slope? No. Consider the following sequence of equations:

$$m = \frac{y_2-y_1}{x_2-x_1} = 1 \times \frac{y_2-y_1}{x_2-x_1} = \frac{-1}{-1} \times \frac{y_2-y_1}{x_2-x_1} = \frac{-1(y_2-y_1)}{-1(x_2-x_1)} = \frac{y_1-y_2}{x_1-x_2}$$

That means $m = \frac{y_2-y_1}{x_2-x_1} = \frac{y_1-y_2}{x_1-x_2}$. Consistently switching the indices 1, and 2 in the slope formula does not change the value of calculated slope.

Example 1: Find the slope of a line passing through (3, 6) and (4, 9)

x_1, y_1	x_2, y_2	
(3, 6)	(4, 9)	$m = \frac{9-6}{4-3} = \frac{3}{1} = \mathbf{3}$
x_1, y_1	x_2, y_2	
(4, 9)	(3, 6)	$m = \frac{6-9}{3-4} = \frac{-3}{-1} = \mathbf{3}$

Example 2: Find the slope of a line passing through (***X₁, Y₁***) and (***X₂, Y₂***). Note we used Bold-Capital-Italicized ***X*** and ***Y*** as the actual coordinates here. Notice the two computed slopes are identical.

x_1, y_1 x_2, y_2

(X_1, Y_1) (X_2, Y_2) $m = \frac{Y_2 - Y_1}{X_2 - X_1}$

x_1, y_1 x_2, y_2

(X_2, Y_2) (X_1, Y_1) $m = \frac{Y_1 - Y_2}{X_1 - X_2} = \frac{-(Y_2 - Y_1)}{-(X_2 - X_1)} = \frac{Y_2 - Y_1}{X_2 - X_1}$

6. What does it mean when either horizontal change = 0 or vertical change = 0 during slope calculation?

When a student calculates slope and has a zero in the numerator or denominator, it is a perfect chance to bring out the meaning of these important special cases:

When the *vertical change is zero*, it literally means as you move along the line, there is *no change in height (or, vertical coordinate remains the same, y = constant)*, as demonstrated by the horizontal orange line with a zero slope.

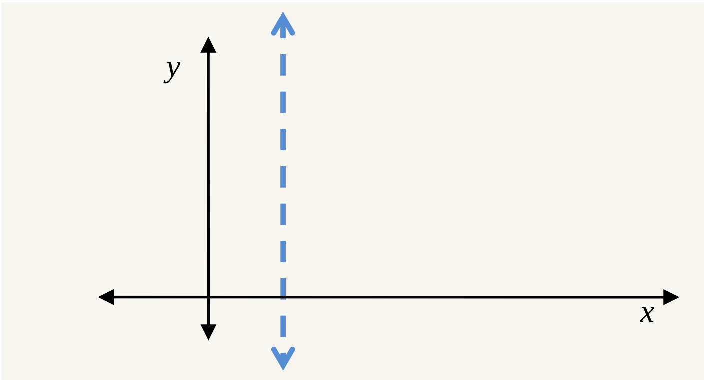
x_1, y_1 x_2, y_2

(6, 9) (4, 9) $m = \frac{9-9}{4-6} = \frac{0}{-2} = 0$

When the *horizontal change is zero*, it literally means as you move along the line, there is *no change in the horizontal coordinate (i.e., x = constant)*, as demonstrated by the vertical blue dashed line. Let's look at an example: applying the slope formula leads to a division by 0 and therefore an undefined slope.

x_1, y_1 x_2, y_2

(2, 5) (2, 4) $m = \frac{4-5}{2-2} = \frac{-1}{\mathbf{0}} =$ → undefined slope



7. What does "no slope" mean?

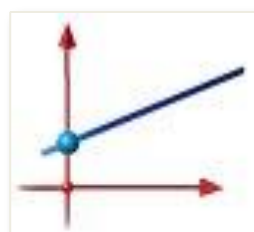
Good question. "No slope" is a misleading term that should not be used in mathematics.

- Horizontal lines have a well-defined slope, namely 0. It is incorrect to say a horizontal line has "no slope."
- Vertical line is a different story. If one applies the definition of slope to a vertical line, it leads to a division by 0. Since we do not know how to handle that, we say the slope of a vertical line is "undefined."

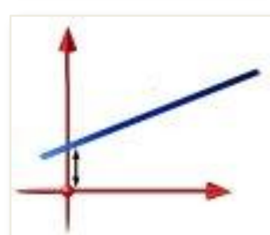
In both cases, we do not use the term "no slope." (See an example of each in question 6.)

8. What is the *y-intercept* of the linear equation y = mx + b? Is it (0, b) or b?

The term "y-intercept" can be used in different contexts here:



(1) When "y-intercept" refers to the **point** (blue ball) where the line intersects the y-axis, then it is a **location** on the x-y plane requiring **two numbers** in the form of an ordered pair, i.e., (0, b).



(2) When "y-intercept" refers to the distance between the origin and where the line intersects the y-axis, as indicated by the black double arrow, then it is a **length** requiring only **one number**, b. It is the "b" in y = mx + b.

So, the y-intercept of a line could mean b or (0, b), depending on the context.

9. Can y = mx + b, where m and b are rational numbers, represents every line on the x-y plane? Concept

No.

1. The vertical line represents an exception. Since its slope is undefined, a vertical line does not have an ***m*** that can be plug into y = ***m***x + b. The equation of a vertical line is in the form x = c, where c is the x-intercept.
2. There are lines on the x-y plane whose slopes (or y-intercepts) are irrational numbers. For example, $= \boldsymbol{\pi} x + 1$, or $y = \sqrt{\mathbf{2}}x - 3$, or $y = x + \sqrt{\mathbf{3}}$. The fractional form of the definition of slope $m = \frac{y_2 - y_1}{x_2 - x_1}$ may leave many students with the false impression that all slopes are fractions, and therefore rational. This is incorrect—x_1, y_1, x_2, y_2 could be irrational themselves and could easily lead to lines with irrational slopes and intercepts.

10. Can y = mx + b, where m and b are real numbers, represents every line on the x-y plane? Concept

No. The vertical line represents an exception. Since its slope is undefined, a vertical line does not have an ***m*** that can be plug into y = ***m***x + b. The equation of a vertical line is in the form x = c, where c is the x-intercept. The form x = c cannot be derived from y = mx + b.

11. How do I write the equation of a line from a word problem? Application

[To be developed]

12. Are there other forms of equations for straight lines?

[To be further developed]

$Ax + By = C$	Standard form
$\frac{x}{a} + \frac{y}{b} = 1$	Intercept/ intercept form
$y - y_1 = m(x - x_1)$	Point-slope form
$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1)$	Two point form

In addition, there are polar form and parametric form.

13. Are the different forms of linear equations all equivalent? Concept

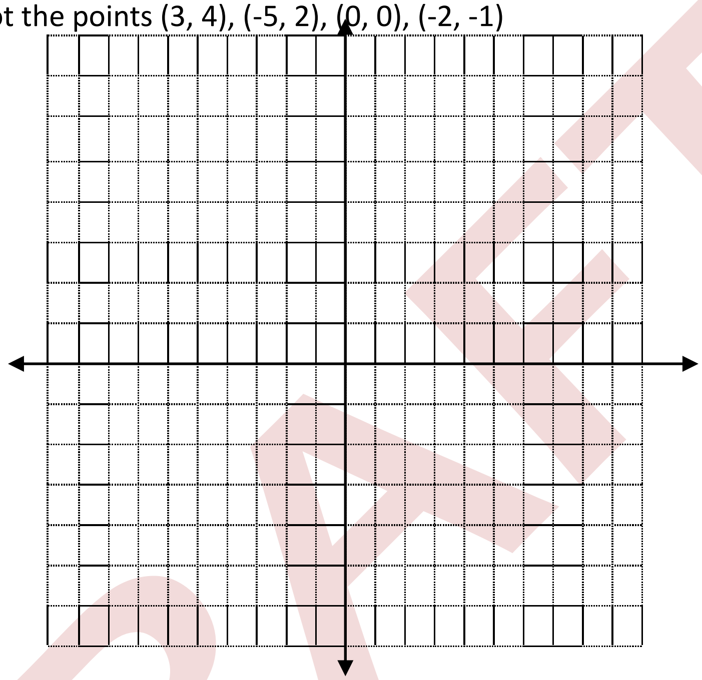
[To be developed]

III. LESSON

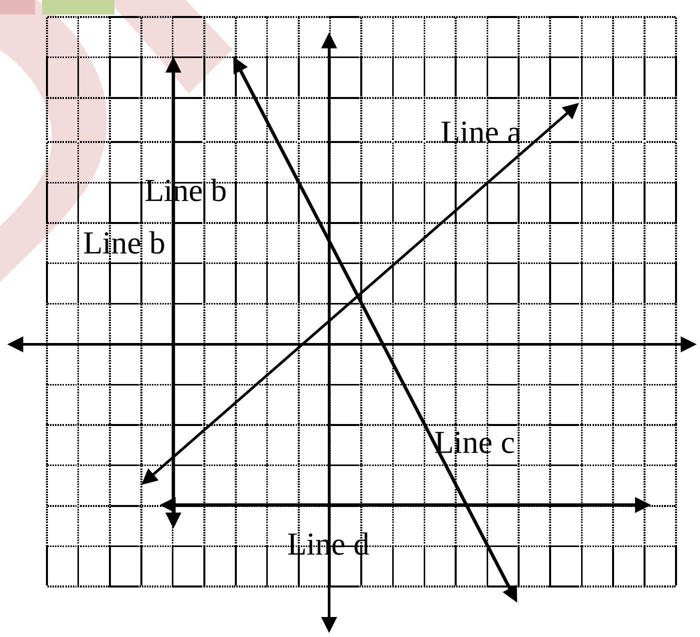
A. Assumptions about what students know and are able to do coming into this lesson

Students should be able to plot points on a coordinate grid. Fluency

Examples: Plot the points (3, 4), (-5, 2), (0, 0), (-2, -1)



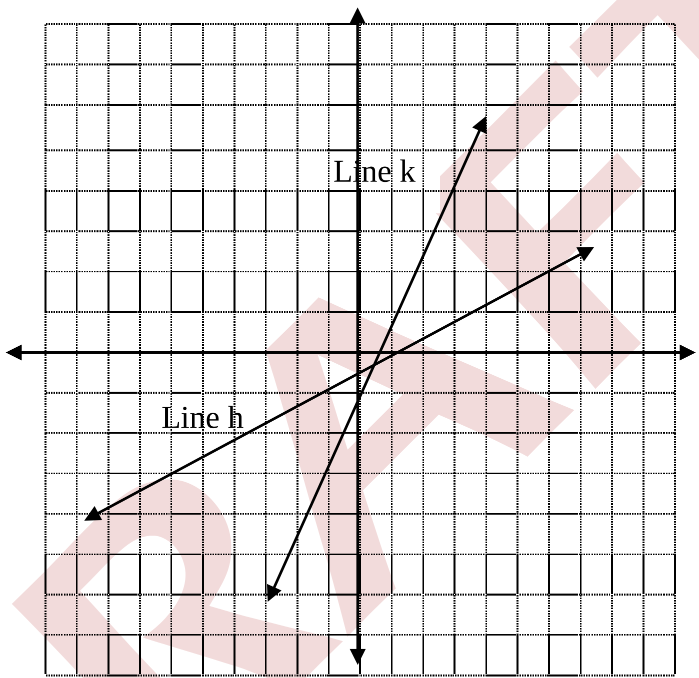
Students should be able to identify whether a given line has a positive, negative, zero, or undefined slope. Example: Which of the following lines have a positive, negative, zero, or undefined slope? Fluency Concept



1. Given any two points, students should be able to find the slope of a line.

Example: What is the slope of a line that passes through (-4, 5) and (-5, 4)? Fluency Concept

2. Students should be able to identify the points of intersection between lines and give the coordinates at those points of intersection.
   Example: What is the point of intersection between *line h* and *line k*? Assume each cell is 1 x 1.



B. Objectives

By the end of the lesson, the students will understand the concepts of slope and y-intercept in a linear equation Concept*, and be able to* derive the equation of a line, Fluency y = mx + b or y = mx when

a. given a slope and the y-intercept of a line.
b. given the slope and a point on the line.
c. given two points on the line.

C. Anticipated Student Preconceptions/ Misconceptions

1. **Students will sometimes read the coordinates of a point incorrectly, starting with the y-coordinate followed by the x-coordinate.** Fluency

   Remedy: Provide the students with additional practice problems on plotting coordinate points and reading the coordinates of given points. Also, remind the students that the coordinates of a point are (x, y) where the first letter comes earlier than the second letter in the alphabet. The students may also need a reminder that the x-axis is horizontal and the y-axis is vertical.

2. **Students may make mistakes when calculating the slope of a line by dividing the horizontal change by the vertical change. Teachers around the country have derived shortcuts to help students remember the correct ratio. Some of these shortcuts are not mathematically sound, and are primarily used in this country, e.g., $\frac{rise}{run}$.**

   Remedy: To curb this problem, it is important to give students practice problems that reinforce how to find the slope of a line. Continued practice will help students remember the slope ratio more readily.

3. **Students sometimes make mistakes when using the formula for slope $\frac{y_1 - y_2}{x_1 - x_2}$ or $\frac{y_2 - y_1}{x_2 - x_1}$ by randomly plugging in the numbers without being aware that the order of the numbers is direction-sensitive. For the numerator, they would plug in $y_1 - y_2$ and then for the denominator they would plug in $x_2 - x_1$, and the sign of the slope would be incorrect.** Fluency MP6

   Remedy: Continued practice for fluency, including drawing the lines, and/or visualizing them, would help eradicate this problem.

4. **When given two points and asked to find the equation of the line, students will incorrectly work through the algorithm of finding the slope and y-intercept without making sense of the line. The students are also likely to make the mistake of thinking that when the denominator is zero, then the slope is zero.** Fluency Concept

   Remedy: Before the students find the slope and y-intercept of the line, they need to either draw a rough sketch of the line or visualize the line and figure out if the slope is positive, negative, zero, or undefined and approximately where the y-intercept would be and then see if the calculations make sense.

5. **Students usually assume that the scales on the x- and y- axes are of the ratio 1:1.** Concept

   Remedy: Students should be exposed to having given points on coordinate grids that have different scales. This means that the x-axis could have increments of one unit while the y-axis has increments of 2 or more or fractional increments.

6. **Students usually think that when their calculations yield zero as the answer, then they have either made a mistake or there is no solution. This could happen every step of the way, when they calculate vertical change, horizontal change, slope, and y-intercept.** Concept

   Remedy: Zero value often offers interesting cases for discussion. (See Advanced Content Knowledge #6.) Students should be provided with information that zero is a number and is a legitimate value. They could also be given examples from real-life situations when zero has tangible meaning, such as a bank statement with a zero balance means that there is no money in the account.

7. **Students have a tendency to think that a straight line means a horizontal or vertical line and not slanted lines**. Concept

   Remedy: Students should be informed that *any two* distinct points on a plane define a straight line.

Instructional tools

MP5 A worksheet with non-numbered grids; a worksheet with numbered grids such that the scales on the x- and y- axes are not a ratio of 1:1; a document camera.

D. Assessments

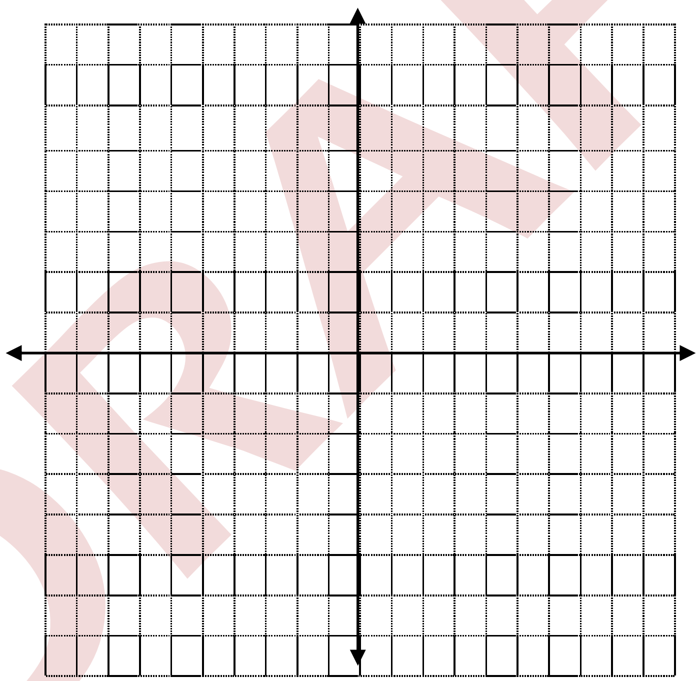
Fluency Concept

Review and discussion problem:

a. Evaluate the equation y = -2x + 4 for x = -4, -2, 0, 3, 5 and fill in the table below.

x					
y					

b. Plot the points on the given coordinate grids.



c. Based on the line you drew in part (b), predict if its slope will be positive, negative, zero, or undefined.

d. Calculate the slope of the line.

e. What is the point of intersection between the line and the y-axis.

E. Lesson Sequence and Description

The following lesson sequence provides a baseline for building foundation skills and concepts. Additional materials should be used to make the practice standards come alive (For example, see a separate document Challenging Problems and Tasks).

Fluency Concept The class discussion starts with the pre-assessment so that the misconceptions that the students may have are corrected. Then the students' attention is drawn to the equation given in the pre-assessment problem and the connection of the numbers in the equation to the calculations found in parts (d) and (e). (e.g., -2 to slope and 4 to the y intercept). The students will then be given two points such us (-3, 4) and (3, -8) and they will plot the points on the x-y grid and then they will find the slope of the line and the point of intersection between this line and the y-axis. MP1 This point is introduced to them as the y-intercept of a line. It is at this point that they can be informed that a line has both the y and x- intercepts but for our purposes, we are to use only the y-intercept for the equation of the line. Based on the pre-assessment example, they are introduced to the general equation of a line, y = mx + b. The initial practice problems should include lines that pass through the origin and those that do not pass through the origin, which will further reinforce the "b" as the point where the line crosses the y-axis. With practice with lines which pass through the origin, the students should be made aware that the y-intercept is equal to zero. Because zero is the addition identity, the sum of any number and zero is that number. Therefore the sum of mx + 0 is simply mx yielding the equation y= mx. The students should be able to have a discussion about the distinguishing parts of the equation. The initial problems of the lesson should be more concrete so that the students would be able to visualize the line followed by more abstract problems so that the students would be able apply the knowledge.

1. Draw a line that passes through (-8, -3) and (2, 2). i. Calculate the slope of the line. ii. Find the y-intercept of the line. iii. Write the equation of the line in slope-intercept form.	2. i. Write the equation of a line in the form y = mx + b whose slope is -3 and the y-intercept is -1. ii. Make a table of results for this graph. iii. Plot the line on an x-y grid.
3. i. Graph a line whose slope is $\frac{1}{3}$ and it passes through (4, 3). ii. Write the equation of the line, in y = mx + b form, in part (i).	4. i. Without graphing, write the equation of a line whose slope is 2 and the y-intercept is -2. ii. Describe the line and state two points it would pass through.
5. Write the equation of a line that passes through (4, -3) and has a slope of $\frac{5}{2}$.	6. Write the equation of a line that passes through (-3, 3) and (3, -3).
7. i. Write the equation whose slope is 0 and it passes through (5, 1). ii. Graph the line. iii. Describe the line.	8. i. Write the equation that passes through (-3, -2) and (-3, 5). ii. Graph the line. iii. Describe the line.

F. Closure

Review outcomes of lesson:

- Students understand what y-intercepts could represent in real-life scenarios. Concept Application
- Students can give a unit rate given two related quantities of different measure. Fluency Concept
- Students understand how real-life situations are modeled with linear relationships and can create a table of equivalent ratios given two related quantities of different measure. Fluency Application
- Students can graph the information from the table of equivalent ratios on an x-y plane. Fluency Concept
- Students can use the graph to predict outcomes based on the rates, assuming the rates are constant. Fluency Concept Application

G. Teacher Reflection

Did the students accomplish the outcomes?
What evidence do I have?
What would I do differently next time?

IV. Worksheets

A. Class Practice

Extension questions for class work or homework:

For these questions, the equation of the line should be expressed in y = mx + b form, where possible. Fluency Concept

1. Write the equation of a line with the given slope and y-intercept:

   a. m = -3; b = -6

   b. m = 0; b = 7

2. Without graphing, write the equation of a line that passes through the two given points:

   a. (3, 4) and (-7, -1)

   b. $\left(\frac{-5}{6}, 3\right)$ and $\left(\frac{5}{9}, -3\right)$

   c. $\left(\frac{-5}{6}, 3\right)$ and $\left(\frac{-5}{6}, -3\right)$

   d. (5, 2) and (-2, 2)

Fluency Concept

3. Find the equation of the line that has the given slope and passes through the given point:

   a. m = 4; (1, 4)

   b. m = 0; (-5, 6)

   c. m = $\frac{5}{7}$; $\left(\frac{14}{25}, \frac{-3}{5}\right)$

   d. m = undefined; (-5, -3)

Application Questions: Fluency Concept Application MP1 MP2 MP3 MP4 MP5 MP6 MP7 MP8

1. Fully describe the line with the equation y = -3x + 6. Give three points that the line passes through.

2. What is the equation of a line whose x-intercept is $\frac{-3}{4}$ and the y-intercept is $\frac{1}{2}$.

3. The equation of a line is Ax + By = C, where A and B are not zero. What is the slope and y-intercept of the line?

4. You make and sell bracelets. You spend $33 for supplies and sell the bracelets for $5.50 each.

   a. Write an equation to model the profit. Use *P* for profit and b for the number of bracelets sold.
   b. Graph the equation.
   c. How many bracelets do you need to sell to cover the cost of the expenses? How would you find this information from the graph?
   d. How many bracelets do you need to sell to make a profit? How would you find this information from the graph?

5.

a. Without graphing, determine which set of points below are collinear and which are not.

Set A: (-3, -2), (0, 4), (1.5, 7)
Set B: (1.25, 1.37), (1.28, 1.48), (1.36, 1.70)

b. What is the equation of the line with collinear points?

6. What is the equation of a line that passes through (-2, -1) and it is
   a. Horizontal
   b. Vertical

7. Write the equation of a line passes through (-2, 3), (2, 5), and (6, k). Find k.

8. Write the equation of a line with an x-intercept of -4, and passes through (2, 6). The line passes through the point (p, 10). Find the value of p.

9. Write the equation of a line that passes through (-1, 3) and is parallel to $3x - y = 4$.

10. A magazine offers an online subscription that allows you to view up to 25 archived articles for free. Any additional archived article will cost a fixed amount per article. To view 30 archived articles, you pay $49.15. To view 33 archived articles, you pay $57.40.
    a. What is the cost of each additional archived article for which you pay a fee?
    b. What is the cost of the magazine subscription?

11. Jamie is ordering tickets for a concert online. There is a processing fee for each order, and the tickets are $52 each. Jamie ordered 5 tickets and the cost was $275.
    a. Determine the processing fee. Write a linear equation to represent the total cost C for t tickets.
    b. Make a table of t and C for at least three other numbers of tickets.
    c. Graph this equation. Predict the cost of 8 tickets.

12. Find the value of the missing coordinate n so that the line will pass through each pair of points, and have the given slope value.
    a. (9, n), (6, 3), slope = $\frac{-1}{3}$
    b. (7, -10), (n, 4), slope = -3

B. Homework (Modify from the Class Practice)

[More to be developed]